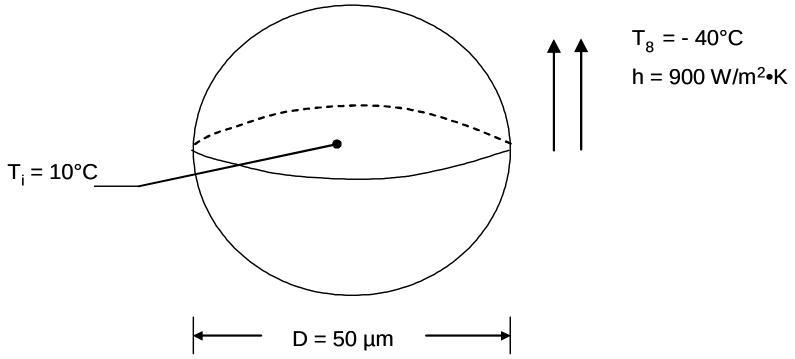
PROBLEM 5.42

KNOWN: Ambient conditions, initial water droplet temperature and diameter.

FIND: Total time to completely freeze the water droplet for (a) droplet solidification at $T_f = 0°C$ and (b) rapid solidification of the droplet at $T_{f,sc}$.

SCHEMATIC:



ASSUMPTIONS: (1) Isothermal particle, (2) Negligible radiation heat transfer, (3) Constant properties.

PROPERTIES: Table A.6, liquid water (T = 0 °C): c_p = 4217 J/kg·K, k = 0.569 W/m·K, ρ = 1000 kg/m^3. Example 1.6: h_{sf} = 334 kJ/kg.

ANALYSIS: We begin by evaluating the validity of the lumped capacitance method by determining the value of the Biot number.

$$B_i = \frac{hL_c}{k} = \frac{hD/3}{k} = \frac{900\ \text{W/m}^2\cdot\text{K} \times 50 \times 10^{-6}\ \text{m}/3}{0.569\ \text{W/m}\cdot\text{K}} = 0.026 << 0.1$$

Hence, the lumped capacitance approach is valid.

Case A: Equilibrium solidification, $T_f = 0°C$.

The solidification process occurs in two steps. The first step involves cooling the drop to T_f = 0°C while the drop is completely liquid. Hence, Equation 5.6 is used where

$A = \pi D^2 = \pi \times (50 \times 10^{-6}\ \text{m})^2 = 7.85 \times 10^{-9}\ \text{m}^2$ and

$V = 4\pi(D/2)^3/3 = 4 \times \pi \times (50 \times 10^{-6}\ \text{m}/2)^3/3 = 65.4 \times 10^{-15}\ \text{m}^3$. Equation 5.6 may be rearranged to yield

$$t_1 = -\frac{\rho Vc}{hA}\ln\left[\frac{T - T_\infty}{T_i - T_\infty}\right] \tag{1}$$

$$= -\frac{1000\ \text{kg/m}^3 \times 65.4 \times 10^{-15}\text{m}^3 \times 4217\ \text{J/kg}\cdot\text{K}}{900\ \text{W/m}^2 \times 7.85 \times 10^{-9}\ \text{m}^2} \times \ln\left[\frac{0 - (-40°C)}{10°C - (-40°C)}\right]$$

$t_1 = 8.7 \times 10^{-3}$ s = 8.7 ms

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The second step involves solidification of the ice, which occurs at $T_f = 0°C$. An energy balance on the droplet yields

$$-E_{out} = \Delta E_{st} \quad \text{or} \quad -hA(T_f - T_\infty)t_2 = \rho V h_{sf}$$

which may be rearranged to provide

$$t_2 = -\frac{\rho V h_{sf}}{hA(T_f - T_\infty)} \qquad (2)$$

$$= \frac{1000\ \text{kg/m}^3 \times 65.4 \times 10^{-15} \text{m}^3 \times 334{,}000\ \text{J/kg}}{900\ \text{W/m}^2 \times 7.85 \times 10^{-9}\ \text{m}^2 \times (0°\text{C} - (-40)°\text{C})} = 77.3 \times 10^{-3}\ \text{s} = 77.3\ \text{ms}$$

The time needed to cool and solidify the particle is

$$t = t_1 + t_2 = 8.7\ \text{ms} + 77.3\ \text{ms} = 86\ \text{ms}$$ <

<u>Case B: Rapid solidification at $T_{f,sc}$.</u>

Using the expression given in the problem statement, the liquid droplet is supercooled to a temperature of $T_{f,sc}$ prior to freezing.

$$T_{f,sc} = -28 + 1.87\ln(50 \times 10^{-6}\ \text{m}) = -36.6°\text{C}$$

The solidification process occurs in multiple steps, the first of which is cooling the particle to $T_{f,sc} = -36.6°C$. Substituting $T = T_{f,sc}$ into Equation 1 yields

$$t_1 = 105 \times 10^{-3}\ \text{s} = 105\ \text{ms}$$

The second step involves rapid solidification of some or all of the supercooled liquid. An energy balance on the particle yields

$$\dot{E}_{st} = 0 = \rho V h_{sf} f = \rho V c(T_f - T_{f,sc}) \qquad (3)$$

where f is the fraction of the mass in the droplet that is converted to ice. Solving the preceding equation for f yields

$$f = \frac{c(T_f - T_{f,sc})}{h_{sf}} = \frac{4217\ \text{J/kg}\cdot\text{K} \times (0°\text{C} - (-36.6°\text{C}))}{334{,}000\ \text{J/kg}} = 0.462$$

Hence, immediately after the rapid solidification, the water droplet is approximately 46% ice and 54% liquid. The time required for the rapid solidification is $t_2 \approx 0$ s.

The third stage of Case B involves the time required to freeze the remaining liquid water, t_3. Equation 2 is modified accordingly to yield

$$t_3 = -\frac{(1-f)\rho V h_{sf}}{hA(T_f - T_\infty)}$$

$$= -\frac{(1 - 0.462) \times 1000\ \text{kg/m}^3 \times 65.4 \times 10^{-15} \text{m}^3 \times 334{,}000\ \text{J/kg}}{900\ \text{W/m}^2 \times 7.85 \times 10^{-9}\ \text{m}^2 \times (0°\text{C} - (-40)°\text{C})} = 42 \times 10^{-3}\ \text{s} = 42\ \text{ms}$$

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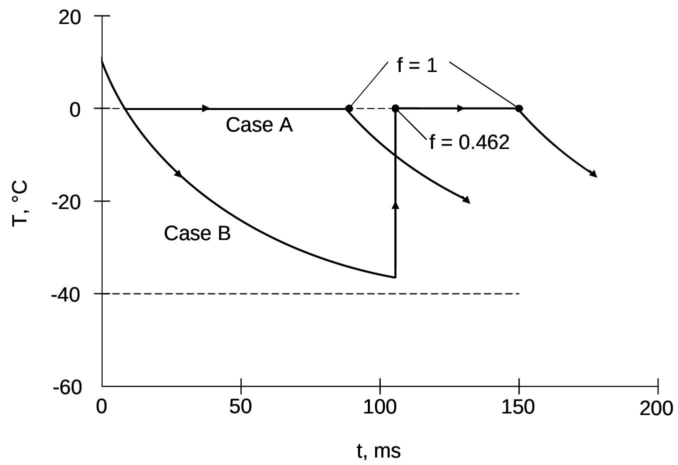
PROBLEM 5.42 (Cont.)

The total time to solidify the particle is

$$t = t_1 + t_2 + t_3 = 105 \text{ ms} + 0 + 42 \text{ s} = 147 \text{ ms} \quad <$$

The temperature histories associated with Case A and Case B are shown in the sketch below.



COMMENTS: (1) Equation 3 may be derived by assuming a reference temperature of $T_f = 0$ °C and a liquid reference state. The energy of the particle prior to the rapid solidification is $E_1 = \rho Vc(T_{f,sc} - T_f)$. The energy of the particle after the rapid solidification is $E_2 = -f\rho Vh_{sf} + (1 - f)\rho Vc(T_f - T_f) = -f\rho Vh_{sf}$. Setting $E_1 = E_2$ yields Equation 3. (2) The average temperature of the supercooled particle is significantly lower than the average temperature of the particle of Case A. Hence, the rate at which the supercooled particle of Case B is cooled by the cold air is, on average, much less than the particle of Case A. Since both particles ultimately reach the same state (all ice at T = 0 °C), it takes longer to completely solidify the supercooled particle. (3) For Case A, the ice particle at T = 0 °C will be a solid sphere, sometimes referred to as *sleet*. For Case B, the rapid solidification will result in a *snowflake*.