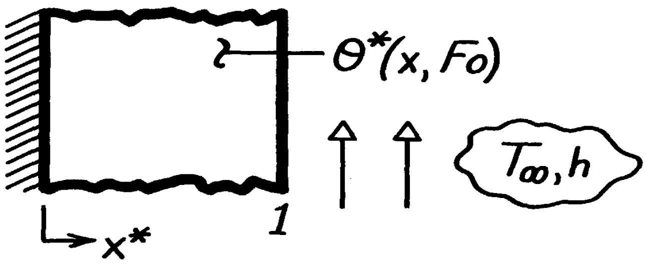
PROBLEM 5.43

KNOWN: Series solution, Eq. 5.42, for transient conduction in a plane wall with convection.

FIND: Midplane ($x^*=0$) and surface ($x^*=1$) temperatures θ^* for Fo=0.1 and 1, using Bi=0.1, 1 and 10 with only the first four eigenvalues. Based upon these results, discuss the validity of the approximate solutions, Eqs. 5.43 and 5.44.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional transient conduction, (2) Constant properties.

ANALYSIS: The series solution, Eq. 5.42a, is of the form,

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp\left(-\zeta_n^2 Fo\right)\cos\left(\zeta_n x^*\right)$$

where the eigenvalues, ζ_n, and the constants, C_n, are from Eqs. 5.39b and 5.39c.

$$\zeta_n \tan \zeta_n = Bi \qquad C_n = 4\sin \zeta_n / \left(2\zeta_n + \sin\left(2\zeta_n\right)\right).$$

The eigenvalues are tabulated in Appendix B.3; note, however, that ζ_1 and C_1 are available from Table 5.1. The values of ζ_n and C_n used to evaluate θ^* are as follows:

Bi	ζ_1	C_1	ζ_2	C_2	ζ_3	C_3	ζ_4	C_4
0.1	0.3111	1.0160	3.1731	-0.0197	6.2991	0.0050	9.4354	-0.0022
1	0.8603	1.1191	3.4256	-0.1517	6.4373	0.0466	9.5293	-0.0217
10	1.4289	1.2620	4.3058	-0.3934	7.2281	0.2104	10.2003	-0.1309

Using ζ_n and C_n values, the terms of θ^*, designated as θ_1^*, θ_2^*, θ_3^* and θ_4^*, are as follows:

Fo=0.1

	Bi=0.1		Bi=1.0		Bi=10	
x^*	0	1	0	1	0	1
θ_1^*	1.0062	0.9579	1.0393	0.6778	1.0289	0.1455
θ_2^*	-0.0072	0.0072	-0.0469	0.0450	-0.0616	0.0244
θ_3^*	0.0001	0.0001	0.0007	0.0007	0.0011	0.0006
θ_4^*	-2.99×10^{-7}	3.00×10^{-7}	2.47×10^{-6}	2.46×10^{-7}	-3.96×10^{-6}	2.83×10^{-6}
θ^*	0.9991	0.9652	0.9931	0.7235	0.9684	0.1705

Continued …

PROBLEM 5.43 (Cont.)

Fo=1						
	Bi=0.1		Bi=1.0		Bi=10	
x^*	0	1	0	1	0	1
θ_1^*	0.9223	0.8780	0.5339	0.3482	0.1638	0.0232
θ_2^*	8.35×10^{-7}	8.35×10^{-7}	-1.22×10^{-5}	1.17×10^{-6}	3.49×10^{-9}	1.38×10^{-9}
θ_3^*	7.04×10^{-20}	-	4.70×10^{-20}	-	4.30×10^{-24}	-
θ_4^*	4.77×10^{-42}	-	7.93×10^{-42}	-	8.52×10^{-47}	-
θ^*	0.9223	0.8780	0.5339	0.3482	0.1638	0.0232

The tabulated results for $\theta^* = \theta^*\left(x^*,\ \mathrm{Bi},\ \mathrm{Fo}\right)$ demonstrate that for Fo=1, the first eigenvalue is sufficient to accurately represent the series. However, for Fo=0.1, three eigenvalues are required for accurate representation.

A more detailed analysis would show that a practical criterion for representation of the series solution by one eigenvalue is Fo > 0.2. For these situations the approximate solutions, Eqs. 5.43 and 5.44, are appropriate. For the midplane, x^*=0, the first two eigenvalues for Fo=0.2 are:

	Fo=0.2	x^*=0	
Bi	0.1	1.0	10
θ_1^*	0.9965	0.9651	0.8389
θ_2^*	-0.00226	-0.0145	-0.0096
θ^*	0.9939	0.9506	0.8293
Error,%	+0.26	+1.53	+1.16

The percentage error shown in the last row of the above table is due to the effect of the second term. For Bi = 0.1, neglecting the second term provides an error of 0.26%. For Bi = 1, the error is 1.53%.

Hence we conclude that the approximate series solutions (with only one eigenvalue) provides systematically high results, but by less than 1.5%, for the Biot number range from 0.1 to 10.