PROBLEM 8.5

KNOWN: The x-momentum equation for fully developed laminar flow in a parallel-plate channel

$$\frac{dp}{dx} = constant = \mu \frac{d^2u}{dy^2}$$

FIND: Following the same approach as for the circular tube in Section 8.1: (a) Show that the velocity profile, *u(y)*, is parabolic of the form

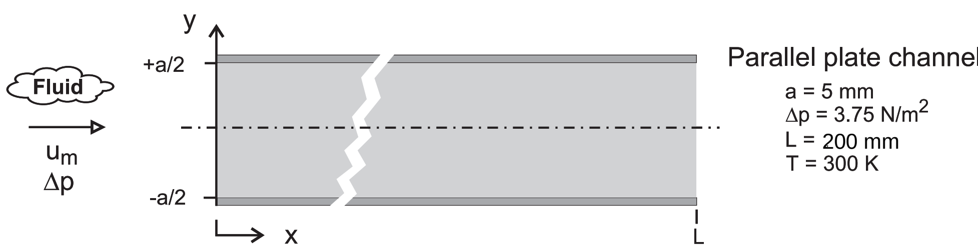
$$u(y) = \frac{3}{2}u_m\left[1 - \frac{y^2}{(a/2)^2}\right]$$

where u_m is the mean velocity expressed as

$$u_m = \frac{a^2}{12\mu}\left(-\frac{dP}{dx}\right)$$

and $-dp/dx = \Delta p/L$ where Δp is the pressure drop across the channel of length *L*; (b) Write the expression defining the friction factor, *f*, using the hydraulic diameter as the characteristic length, D_h; What is the hydraulic diameter for the parallel-plate channel? (c) The friction factor is estimated from the expression $f = C/Re_{D_h}$ where C depends upon the flow cross-section as shown in Table 8.1; What is the coefficient C for the parallel-plate channel ($b/a \rightarrow \infty$) ? (d) Calculate the mean air velocity and the Reynolds number for air at atmospheric pressure and 300 K in a parallel-plate channel with separation of 5 mm and length of 100 mm subjected to a pressure drop of Δp = 3.75 N/m^2; Is the assumption of fully developed flow reasonable for this application? If not, what effect does this have on the estimate for u_m?

SCHEMATIC:



ASSUMPTIONS: (1) Fully developed laminar flow, (2) Parallel-plate channel, a << b.

PROPERTIES: *Table A-4*, Air (300 K, 1 atm): $\mu = 184.6 \times 10^{-7}$ N·s/m^2, $\nu = 15.89 \times 10^{-6}$ m^2/s.

ANALYSIS: (a) The x-momentum equation for fully developed laminar flow is

$$\mu\left(\frac{d^2u}{dy^2}\right) = \frac{dp}{dx} = \text{constant} \qquad (1)$$

Since the longitudinal pressure gradient is constant, separate variables and integrate twice,

$$\frac{d}{dy}\left(\frac{du}{dy}\right) = \frac{1}{\mu}\left(\frac{dp}{dx}\right) \qquad \frac{du}{dy} = \frac{1}{\mu}\left(\frac{dp}{dx}\right)y + C_1$$

$$u = \frac{1}{2\mu}\left(\frac{dp}{dx}\right)y^2 + C_1 y + C_2$$

Continued …

PROBLEM 8.5 (Cont.)

The integration constants are determined from the boundary conditions,

$$\left.\frac{du}{dy}\right|_{y=0} = 0 \qquad\qquad u(a/2) = 0$$

to find

$$C_1 = 0 \qquad\qquad C_2 = -\frac{1}{2\mu}\left(\frac{dp}{dx}\right)(a/2)^2$$

giving

$$u(y) = -\frac{(a/2)^2}{2\mu}\left(\frac{dp}{dx}\right)\left[1-\frac{y^2}{(a/2)^2}\right] \tag{2}$$

The mean velocity is

$$u_m = \frac{2}{a}\int_0^{a/2} u(y)\,dy = -\frac{2}{a}\frac{(a/2)^2}{2\mu}\left(\frac{dp}{dx}\right)\left[y-\frac{y^3/3}{(a/2)^2}\right]_0^{a/2}$$

$$u_m = \frac{a^2}{12\mu}\left(-\frac{dp}{dx}\right) \tag{3}$$

Substituting Eq. (3) for dp/dx into Eq. (2) find the velocity distribution in terms of the mean velocity

$$u(y) = \frac{3}{2}u_m\left[1-\frac{y^2}{(a/2)^2}\right] \qquad < \tag{4}$$

(b) The friction factor follows from its definition, Eq. 8.16,

$$f = \frac{-(dp/dx)D_h}{\rho \cdot u_m^2/2} \tag{5}$$

where the hydraulic diameter for the channel using Eq. 8.66 is

$$D_h = \frac{4\cdot A_c}{P} = \frac{4(a\times b)}{2(a+b)} = 2a \qquad < \tag{6}$$

since a << b.

(c) Substituting for the pressure gradient, Eq. (3), and rearranging, find using Eq. (6),

$$f = \frac{u_m}{a^2/12\mu}\frac{D_h}{\rho u_m^2/2} = \frac{96}{u_m D_h/\nu} = \frac{96}{Re_{D_h}} \qquad < \tag{7}$$

where the Reynolds number is

$$Re_{D_h} = u_m D_h/\nu \tag{8}$$

Continued …

This result is in agreement with Table 8.1 for the cross-section with b/a $\rightarrow \infty$ where

$$C = 96. \qquad <$$

(d) For the conditions shown in the schematic, with air properties evaluated at 300 K, using Eqs. (3) and (8), find

$$u_m = \frac{(0.005\text{m})^2}{12 \times 184.6 \times 10^{-7}\,\text{N}\cdot\text{s/m}^2}\left(\frac{3.75\,\text{N/m}^2}{0.200\text{m}}\right) = 2.12\,\text{m/s}$$

$$Re_D = \frac{2.12\,\text{m/s} \times 2 \times 0.005\text{m}}{15.89 \times 10^{-6}\,\text{m}^2/\text{s}} = 1332$$

The flow is laminar since $Re_{Dh} < 2300$, and from Eq. 8.3, the laminar entry length is

$$\left.\left(\frac{x_{fd,h}}{D_h}\right)\right|_{lam} = 0.05\,Re_{Dh}$$

$$x_{fd,h} = 2 \times 0.005\text{m} \times 0.05 \times 1332 = 0.67\text{m}$$

We conclude that the flow is not fully developed, and the friction factor in the entry region will be higher than for fully developed conditions. Hence, for the same pressure drop, the mean velocity will be less than our estimate.