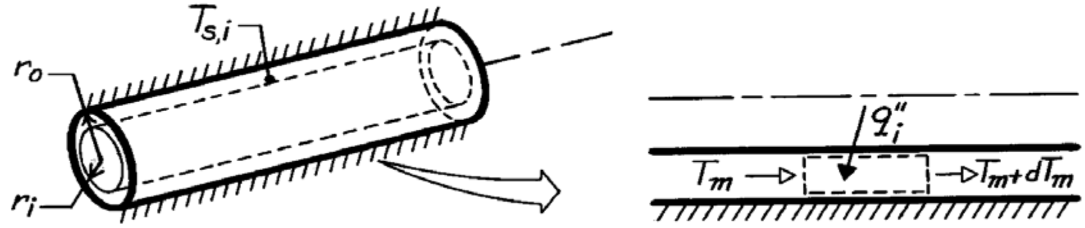
PROBLEM 8.95

KNOWN: Inner and outer tube surface conditions for an annulus.

FIND: (a) Velocity profile, (b) Temperature profile and expression for inner surface Nusselt number.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Laminar, fully developed flow, (3) Uniform heat flux at inner surface, (4) Adiabatic outer surface, (5) Constant properties, (6) Applicability of Eq. 8.34.

ANALYSIS: (a) From Section 8.1.3, the general solution to Eq. 8.12, which also applies to annular flow as represented in Figure 8.11, is

$$u(r) = \frac{1}{\mu}\left(\frac{dp}{dx}\right)\frac{r^2}{4} + C_1\,\ell n\, r + C_2.$$

Applying the boundary conditions,

$$u(r_i) = 0 \qquad 0 = \frac{1}{\mu}\left(\frac{dp}{dx}\right)\frac{r_i^2}{4} + C_1\,\ell n r_i + C_2$$

$$u(r_o) = 0 \qquad 0 = \frac{1}{\mu}\left(\frac{dp}{dx}\right)\frac{r_o^2}{4} + C_1\,\ell n r_o + C_2.$$

Hence,

$$C_1 = \frac{\frac{1}{\mu}\left(\frac{dp}{dx}\right)\left(\frac{r_o^2}{4} - \frac{r_i^2}{4}\right)}{\ell n\, r_i / r_o} \qquad C_2 = -\frac{1}{\mu}\left(\frac{dp}{dx}\right)\frac{r_o^2}{4} - \frac{1}{\mu}\left(\frac{dp}{dx}\right)\left(\frac{r_o^2}{4} - \frac{r_i^2}{4}\right)\frac{\ell n\, r_o}{\ell n\,(r_i / r_o)}$$

and the velocity distribution is

$$u(r) = \frac{1}{\mu}\left(\frac{dp}{dx}\right)\left(\frac{r^2}{4} - \frac{r_o^2}{4}\right) + \frac{1}{\mu}\left(\frac{dp}{dx}\right)\left(\frac{r_o^2}{4} - \frac{r_i^2}{4}\right)\frac{\ell n\, r}{\ell n\,(r_i / r_o)}$$

$$-\frac{1}{\mu}\left(\frac{dp}{dx}\right)\left(\frac{r_o^2}{4} - \frac{r_i^2}{4}\right)\frac{\ell n\, r_o}{\ell n\,(r_i / r_o)}$$

$$u(r) = -\frac{r_o^2}{4\mu}\left(\frac{dp}{dx}\right)\left[1 - (r/r_o)^2 + \frac{(r_i / r_o)^2 - 1}{\ell n\,(r_i / r_o)}\ell n(r/r_o)\right]. \qquad (1) \quad <$$

(b) For fully developed conditions with uniform surface heat flux,

$$v = 0 \qquad \partial T/\partial x = dT_m / dx = \text{const.}$$

Continued …

Hence, from Eq. 8.48, which also applies for annular flow,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\frac{u}{\alpha}\frac{dT_m}{dx}$$

Substituting the velocity distribution, with

$$C_1=-\frac{r_o^2}{4\mu}\left(\frac{dp}{dx}\right) \qquad C_2=\frac{\left(r_i/r_o\right)^2-1}{\ell n\left(r_i/r_o\right)} \tag{2}$$

it follows that $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\frac{C_1}{\alpha}\frac{dT_m}{dx}\left[1-\left(r/r_o\right)^2+C_2\,\ell n\left(r/r_o\right)\right].$

$$r\frac{\partial T}{\partial r}=\frac{C_1}{\alpha}\frac{dT_m}{dx}\int\left[r-\frac{r^3}{r_o^2}+C_2 r\ell n\frac{r}{r_o}\right]dr+C_3$$

$$\frac{\partial T}{\partial r}=\frac{C_1}{\alpha}\frac{dT_m}{dx}\left[\frac{r}{2}-\frac{r^3}{4r_o^2}+C_2\left(\frac{r}{2}\ell n\frac{r}{r_o}-\frac{r}{4}\right)\right]+\frac{C_3}{r}$$

and the temperature distribution is

$$T(r)=\frac{C_1}{\alpha}\frac{dT_m}{dx}\left[\frac{r^2}{4}-\frac{r^4}{16\,r_o^2}+C_2\left(\frac{r^2}{4}\ell n\frac{r}{r_o}-\frac{r^2}{4}\right)\right]+C_3\,\ell nr+C_4. \tag{3} <$$

From the requirement that $q''_o=0$, it follows that $\left.\partial T/\partial r\right)_{r_o}=0$. Hence,

$$\frac{C_1}{\alpha}\frac{dT_m}{dx}\left[\frac{r_o}{2}-\frac{r_o}{4}+C_2\left(-\frac{r_o}{4}\right)\right]+\frac{C_3}{r_o}=0$$

$$C_3=\frac{C_1}{\alpha}\frac{dT_m}{dx}\frac{r_o^2}{4}\left(C_2-1\right). \tag{4} <$$

From the condition that $T(r_i) = T_{s,i}$, it follows that

$$C_4=T_{s,i}-\frac{C_1}{\alpha}\frac{dT_m}{dx}\left[\frac{r_i^2}{4}-\frac{r_i^4}{16\,r_o^2}+C_2\left(\frac{r_i^2}{4}\ell n\frac{r_i}{r_o}-\frac{r_i^2}{4}\right)\right]+C_3\,\ell nr_i. \tag{5} <$$

From Eqs. 8.67 and 8.69, the inner surface Nusselt number is

$$Nu_i=\frac{h_i\,D_h}{k}=\frac{q''_i\,D_h}{k\left(T_{s,i}-T_m\right)}$$

where $D_h = 2(r_o - r_i)$. To obtain a workable form of Nu_i, the mean temperature T_m must be evaluated. This may be done by substituting Eqs. (1) and (3) into Eq. 8.26 and evaluating u_m by substituting Eq. (1) into Eq. 8.8. Since the integrations are long and tedious, they are not provided.

COMMENTS: From an energy balance performed for a differential control volume in the annular region, $dT_m/dx=2r_iq''_i/\rho\,c_p\,u_m\left(r_o^2-r_i^2\right)$.