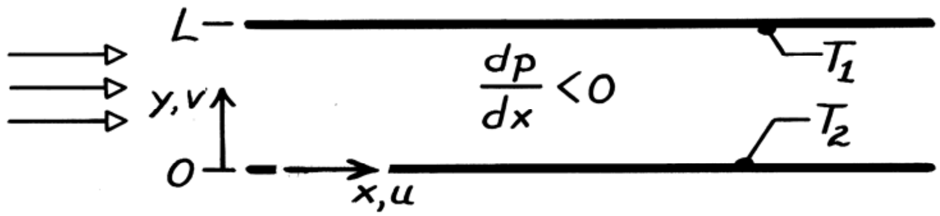
PROBLEM 6S.10

KNOWN: Steady, incompressible, laminar flow between infinite parallel plates at different temperatures.

FIND: (a) Form of continuity equation, (b) Form of momentum equations and velocity profile. Relationship of pressure gradient to maximum velocity, (c) Form of energy equation and temperature distribution. Heat flux at top surface.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional flow (no variations in z) between infinite, parallel plates, (2) Negligible body forces, (3) No internal energy generation, (4) Incompressible fluid with constant properties.

ANALYSIS: (a) For two-dimensional, steady conditions, the continuity equation is

$$\frac{\partial(\rho\, u)}{\partial\, x}+\frac{\partial(\rho\, v)}{\partial\, y}=0.$$

Hence, for an incompressible fluid (constant ρ) in parallel flow (v = 0),

$$\frac{\partial\, u}{\partial\, x}=0. \qquad <$$

The flow is fully developed in the sense that, irrespective of y, u is independent of x.

(b) With the above result and the prescribed conditions, the momentum equations reduce to

$$0=-\frac{\partial\, p}{\partial\, x}+\mu\frac{\partial^{2} u}{\partial\, y^{2}} \qquad 0=-\frac{\partial\, p}{\partial\, y} \qquad <$$

Since p is independent of y, ∂p/∂x = dp/dx is independent of y and

$$\mu\frac{\partial^{2} u}{\partial\, y^{2}}=\mu\frac{d^{2}u}{dy^{2}}=\frac{dp}{dx}.$$

Since the left-hand side can, at most, depend only on y and the right-hand side is independent of y, both sides must equal the same constant C. That is,

$$\mu\frac{d^{2}u}{dy^{2}}=C.$$

Hence, the velocity distribution has the form

$$u(y)=\frac{C}{2\mu}y^{2}+C_{1}y+C_{2}.$$

Using the boundary conditions to evaluate the constants,

$$u(0)=0 \quad\rightarrow\quad C_{2}=0 \quad\text{and}\quad u(L)=0 \quad\rightarrow\quad C_{1}=-CL/2\mu.$$

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The velocity profile is $u(y) = \frac{C}{2\mu}\left(y^2 - Ly\right).$

The profile is symmetric about the midplane, in which case the maximum velocity exists at y = L/2. Hence,

$$u(L/2) = u_{max} = \frac{C}{2\mu}\left[-\frac{L^2}{4}\right] \qquad \text{or} \qquad u_{max} = -\frac{L^2}{8\mu}\frac{dp}{dx}.$$ <

(c) For fully developed thermal conditions, $(\partial T/\partial x) = 0$ and temperature depends only on y. Hence with v = 0, $\partial u/\partial x = 0$, and the prescribed assumptions, the energy equation becomes

$$\rho\, u \frac{\partial\, i}{\partial\, x} = k\frac{d^2T}{dy^2} + u\frac{dp}{dx} + \mu\left[\frac{du}{dy}\right]^2.$$

With $i = e + p/\rho$, $\frac{\partial\, i}{\partial\, x} = \frac{\partial\, e}{\partial\, x} + \frac{1}{\rho}\frac{dp}{dx}$ where $\frac{\partial\, e}{\partial\, x} = \frac{\partial\, e}{\partial\, T}\frac{\partial\, T}{\partial\, x} + \frac{\partial\, e}{\partial\, \rho}\frac{\partial\, \rho}{\partial\, x} = 0.$

Hence, the energy equation becomes $0 = k\frac{d^2T}{dy^2} + \mu\left[\frac{du}{dy}\right]^2.$ <

With $du/dy = (C/2\mu)(2y - L)$, it follows that

$$\frac{d^2T}{dy^2} = -\frac{C^2}{4k\mu}\left(4y^2 - 4Ly + L^2\right).$$

Integrating twice,

$$T(y) = -\frac{C^2}{4k\mu}\left[\frac{y^4}{3} - \frac{2Ly^3}{3} + \frac{L^2y^2}{2}\right] + C_3y + C_4$$

Using the boundary conditions to evaluate the constants,

$$T(0) = T_2 \quad\rightarrow\quad C_4 = T_2 \quad \text{and} \quad T(L) = T_1 \quad\rightarrow\quad C_3 = \frac{C^2L^3}{24k\mu} + \frac{(T_1 - T_2)}{L}.$$

Hence, $T(y) = T_2 + \left[\frac{y}{L}\right](T_1 - T_2) - \frac{C^2}{4k\mu}\left[\frac{y^4}{3} - \frac{2Ly^3}{3} + \frac{L^2y^2}{2} - \frac{L^3y}{6}\right].$ <

From Fourier's law,

$$q''(L) = -k\left.\frac{\partial\, T}{\partial\, y}\right|_{y=L} = \frac{k}{L}(T_2 - T_1) + \frac{C^2}{4\mu}\left[\frac{4}{3}L^3 - 2L^3 + L^3 - \frac{L^3}{6}\right]$$

$$q''(L) = \frac{k}{L}(T_2 - T_1) + \frac{C^2L^3}{24\mu}.$$ <

COMMENTS: The third and second terms on the right-hand sides of the temperature distribution and heat flux, respectively, represents the effects of viscous dissipation. If C is large (due to large μ or u_{max}), viscous dissipation is significant. If C is small, conduction effects dominate.