CHAPTER 8 Linear Algebra: Matrix Eigenvalue Problems

Prerequisite for this chapter is some familiarity with the notion of a matrix and with the two algebraic operations for matrices. Otherwise the chapter is independent of Chap. 7, so that it can be used for teaching eigenvalue problems and their applications, without first going through the material in Chap. 7.

SECTION 8.1. The Matrix Eigenvalue Problem. Determining Eigenvalues and Eigenvectors, page 323

Purpose. To familiarize the student with the determination of eigenvalues and eigenvectors of real matrices and to give a first impression of what one can expect (multiple eigenvalues, complex eigenvalues, etc.).

Main Content, Important Concepts

Eigenvalue, eigenvector

Determination of eigenvalues from the characteristic equation

Determination of eigenvectors

Algebraic and geometric multiplicity, defect

Comments on Content

To maintain undivided attention on the basic concepts and techniques, all the examples in this section are formal, and typical applications are put into a separate section (Sec. 8.2).

The distinction between the algebraic and geometric multiplicity is mentioned in this early section, and the idea of a *basis of eigenvectors* ("**eigenbasis**") could perhaps be mentioned briefly in class, whereas a thorough discussion of this in a later section (Sec. 8.4) will profit from the increased experience with eigenvalue problems, which the student will have gained at that later time.

The possibility of *normalizing* any eigenvector is mentioned in connection with Theorem 2, but this will be of greater interest to us only in connection with orthonormal or unitary systems (Secs. 8.4 and 8.5).

In our present work we find eigen*values* first and are then left with the much simpler task of determining corresponding eigen*vectors.* Numeric work (Secs. 20.62–20.9) may proceed in the opposite order, but to mention this here would perhaps just confuse the student.

Further Comments on Examples and Theorems

The simple examples should give the student a first impression of what to expect.

In particular, Example 4 shows that eigenvalue problems lead to work in complex, even if the matrices are real. This is an important point to emphasize.

Theorem 1 shows that for matrices, in contrast to differential equations, eigenvalue problems involve no existence questions since the existence of an eigenvalue is always guaranteed.

Theorems 2 and 3 concern the notion of eigenspace and the invariance of eigenvalues under transposition.

Comments on Problems

Problems 1–16 involve straightforward calculations to gain skill and an understanding of the concepts involved. Sidetracking attention by solving cubic or higher-order equations is avoided.

Problems 17–20 illustrate simple applications to analytic geometry. Actual applications of eigenvalue problems follow in the next section, as has been mentioned before.

Problems 21–25 illustrate some important simple facts.

SOLUTIONS TO PROBLEM SET 8.1, page 329

1. Eigenvalues are: $3/2$ and 3 and the corresponding eigenvectors are $[1, 0]^T$, and $[0, 1]^T$ respectively.

2. This zero matrix, like any square zero matrix, has the eigenvalue 0. The algebraic multiplicity and geometric multiplicity are both equal to 2, and we can choose the basis $[1 \quad 0]^\mathsf{T}$, $[0 \quad 1]^\mathsf{T}$.

3. Eigenvalues are 0 and -3, and the corresponding eigenvectors are $[2/3, 1]^T$, and $[1/3, 1]^T$ respectively.

4. Eigenvalues are 5 and 0 and eigenvectors $[1 \quad 2]^\mathsf{T}$ and $[-2 \quad 1]^\mathsf{T}$, respectively. The matrix is symmetric, and for such a matrix it is typical that the eigenvalues are real and the eigenvectors orthogonal. Also, make the students aware of the fact that 0 can very well be an eigenvalue.

5. Eigenvalues are $4i$ and $-4i$, and the corresponding eigenvectors are $[-i, 1]^T$ and $[i, 1]^T$ respectively.

6. Eigenvalues are 1 and 3 and eigenvectors $[1 \quad 0]^\mathsf{T}$ and $[1 \quad 1]^\mathsf{T}$, respectively. Note that for such a triangular matrix, the main diagonal entries are still the eigenvalues (as for a diagonal matrix; cf. Prob. 1), but the eigenvectors are no longer orthogonal.

7. The matrix has a repeated eigenvalue of 0 with eigenvectors $[1, 0]^T$, and $[0, 0]^T$.

8. Eigenvalues: $a \pm \sqrt{-k}$; Eigenvectors: $[1/\sqrt{-k}, 1]^T$ and $[-1/\sqrt{-k}, 1]^T$, respectively.

9. Eigenvalues are 0.20 ± 0.40, and the eigenvectors are $[i, 1]^T$ and $[-i, 1]^T$ respectively.

10. The characteristic equation is

$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0.$$

Solutions (eigenvalues) are $\lambda = \cos\theta \pm i\sin\theta$. Eigenvectors are obtained from

$$(\lambda - \cos\theta)x_1 + (\sin\theta)x_2 = (\sin\theta)(\pm i x_1 + x_2) = 0,$$

say, $x_1 = 1, x_2 = \mp i$.

Note that this matrix represents a rotation through an angle θ, and this linear transformation preserves no real direction in the x_1x_2-plane, as would be the case if the eigenvectors were positive real. This explains why these vectors must be complex.

11. Eigenvalues: 4, 1, 7, Eigenvectors are: $[-1/2, 1, 1]^T$, $[1, -1/2, 1]^T$, $[-2, -2, 1]^T$.

12. 3, $[1 \quad 0 \quad 0]^\mathsf{T}$; 4, $[5 \quad 1 \quad 0]^\mathsf{T}$; 1, $[7, \quad -4 \quad 2]^\mathsf{T}$

13. Repeated eigenvalue 2. Eigenvectors: $[2, -2, 1]^T$, $[0, 0, 0]^T$, $[0, 0, 0]^T$.

14. Develop the characteristic determinant by the second row, obtaining

$$(\tfrac{1}{2} - \lambda)[(2 - \lambda)(4 - \lambda) + 1] = (\tfrac{1}{2} - \lambda)(\lambda - 3)^2.$$

Eigenvectors for the eigenvalues $\frac{1}{2}$ and 3 are $[0 \quad 1 \quad 0]^\mathsf{T}$ and $[-1 \quad 0 \quad 1]^\mathsf{T}$, respectively, and we get no basis for R^3.

16. The indicated division of the characteristic polynomial gives

$$(\lambda^4 - 22\lambda^2 + 24\lambda + 45)/(\lambda - 3)^2 = \lambda^2 + 6\lambda + 5.$$

The eigenvalues and eigenvectors are

$$\begin{aligned} &\lambda_1 = 3, && [1 \quad 1 \quad 1 \quad 1]^\mathsf{T} \text{ with a defect of 1} \\ &\lambda_2 = -1 && [3 \quad -1 \quad 1 \quad 1]^\mathsf{T} \\ &\lambda_3 = -5, && [-11 \quad 1 \quad 5 \quad 1]^\mathsf{T}. \end{aligned}$$

18. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; **(a)** 1, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$; **(b)** -1, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. **(a)** Any point of the x_1-axis is mapped onto itself. **(b)** Any point $(0, x_2)$ on the x_2-axis is mapped onto $(0, -x_2)$, so that $[0 \quad x_2]^\mathsf{T}$ is an eigenvector corresponding to $\lambda = -1$.

20. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The eigenvalue 1 with eigenvectors $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ (which span the plane $x_2 = x_1$) indicates that every point in the plane $x_2 = x_1$ is mapped onto itself. The other eigenvalue 0 with eigenvector $[1 \quad -1 \quad 0]^\mathsf{T}$ indicates that any point on the line $x_2 = -x_1, x_3 = 0$ (which is perpendicular to the plane $x_2 = x_1$) is mapped onto the origin. The student should perhaps make a sketch to see what is going on geometrically.

24. By Theorem 1 in Sec. 7.8 the inverse exists if and only if $\det \mathbf{A} \neq 0$. On the other hand, from the product representation

$$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

of the characteristic polynomial we obtain

$$\det \mathbf{A} = (-1)^n(-\lambda_1)(-\lambda_2)\cdots(-\lambda_n) = \lambda_1\lambda_2\cdots\lambda_n.$$

Hence $\mathbf{A}^{-1}$ exists if and only if 0 is not an eigenvalue of $\mathbf{A}$.

Furthermore, let $\lambda \neq 0$ be an eigenvalue of $\mathbf{A}$. Then

$$\mathbf{Ax} = \lambda\mathbf{x}.$$

Multiply this by $\mathbf{A}^{-1}$ from the left:

$$\mathbf{A}^{-1}\mathbf{Ax} = \lambda\mathbf{A}^{-1}\mathbf{x}.$$

Now divide by λ:

$$\frac{1}{\lambda}\mathbf{x} = \mathbf{A}^{-1}\mathbf{x}.$$

SECTION 8.2. Some Applications of Eigenvalue Problems, page 329

Purpose. Matrix eigenvalue problems are of greatest importance in physics, engineering, geometry, etc., and the applications in this section and in the problem set are supposed to give the student at least some impression of this fact.

Main Content

Applications of eigenvalue problems in

Elasticity theory (Example 1)

Probability theory (Example 2)

Biology (Example 3)

Mechanical vibrations (Example 4)

Short Courses. Of course, this section can be omitted, for reasons of time, or one or two of the examples can be considered quite briefly.

Comments on Content

The examples in this section have been selected from the viewpoint of modest prerequisites, so that not too much time will be needed to set the scene.

Example 4 illustrates why real matrices can have complex eigenvalues (as mentioned before, in Sec. 8.1), and why these eigenvalues are physically meaningful. (For students familiar with systems of ODEs, one can easily pick further examples from Chap. 4.)

Comments on Problems

Problems 1–12 are similar to the applications shown in the examples of the text.

Problems 13–15 show an interesting application of eigenvalue problems to production, typical of various other applications of eigenvalue theory in economics included in various textbooks in economic theory.

SOLUTIONS TO PROBLEM SET 8.2, page 333

1. Eigenvalue and eigenvectors are -1, $[-1, 1]^T$ and 2, $[1, 1]^T$. The eigenvectors are orthogonal.

2. Eigenvalues and eigenvectors are 1.6, $[1 \quad -1]^\mathsf{T}$ and 2.4, $[1 \quad 1]^\mathsf{T}$. These vectors are orthogonal, as is typical of a symmetric matrix. Directions are $-45°$ and $45°$, respectively.

3. Eigenvalues 3, -3 and eigenvectors $[\sqrt{2}, 1]^T$ and $[-1/\sqrt{2}, 1]^T$ respectively.

4. Extension factors $9 + 2\sqrt{5} = 13.47$ and $9 - 2\sqrt{5} = 4.53$ in the directions given by $[1 \quad 2 + \sqrt{5}]^\mathsf{T}$ and $[1 \quad 2 - \sqrt{5}]^\mathsf{T}$ (76.7° and $-13.3°$, respectively).

6. 2, $[1 \quad 1]^\mathsf{T}$; $\frac{1}{2}$, $[1 \quad -1]^\mathsf{T}$; directions 45° and $-45°$, respectively.

7. Eigenvector $[2.5, 1]^T$ with eigenvalue 1.

8. $[1 \quad 1 \quad 1]^\mathsf{T}$, as could also be seen without calculation because $\mathbf{A}$ has row sums equal to 1, which would not be the case in general.

9. Eigenvector $[-0.2, -0.4, 1]$ with eigenvalue 1.

10. Growth rate 3. The characteristic polynomial is $\frac{1}{4}(x - 3)(2x + 5)(2x + 1)$ which gives the remaining two eigenvalues as 2.5 and 0.5. The sum of all the eigenvalues is the trace which is zero. Note that the growth rate is not that sensitive to the elements of the matrix. Working with two decimal digits still retains the intrinsic characteristic of the problem.

11. Growth rate is 4. Characteristic polynomial is $(x - 4)(x + 1)(x + 3)$.

12. Growth rate 1.3748. The other eigenvalues are complex or negative and are not needed. The sum of all eigenvalues equals the trace, that is, 0, except for a roundoff error. This 4×4 Leslie matrix corresponds to a classification of the population into four classes.

14. $\mathbf{A}$ has the same eigenvalues as $\mathbf{A}^\mathsf{T}$, and $\mathbf{A}^\mathsf{T}$ has row sums 1, so that it has the eigenvalue 1 with eigenvector $\mathbf{x} = [1 \cdots 1]^\mathsf{T}$.

Leontief is a leader in the development and application of quantitative methods in empirical economical research, using genuine data from the economy of the United States to provide, in addition to the "closed model" of Prob. 13 (where the producers consume the whole production), "open models" of various situations of production and consumption, including import, export, taxes, capital gains and losses, etc. See W. W. Leontief, *The Structure of the American Economy 1919–1939* (Oxford: Oxford University Press, 1951). H. B. Cheney and P. G. Clark, *Interindustry Economics* (New York: Wiley, 1959).

16. This follows by comparing the coefficient of λ^{n-1} in the development of the characteristic determinant $D(\lambda)$ with that obtained from the product representation.

18. The first statement follows from

$$\mathbf{Ax} = \lambda\mathbf{x}, \quad (k\mathbf{A})\mathbf{x} = k(\mathbf{Ax}) = k(\lambda\mathbf{x}) = (k\lambda)\mathbf{x},$$

the second by induction and multiplication of $\mathbf{A}^k\mathbf{x}_j = \lambda_j^k\mathbf{x}_j$ by $\mathbf{A}$ from the left.

20. $\det(\mathbf{L} - \lambda\mathbf{I}) = -\lambda^3 + l_{12}l_{21}\lambda + l_{13}l_{21}l_{32} = 0$. Hence $\lambda \neq 0$. If all three eigenvalues are real, at least one is positive since trace $\mathbf{L} = 0$. The only other possibility is $\lambda_1 = a + ib$, $\lambda_2 = a - ib$, λ_3 real (except for the numbering of the eigenvalues). Then $\lambda_3 > 0$ because

$$\lambda_1\lambda_2\lambda_3 = (a^2 + b^2)\lambda_3 = \det \mathbf{L} = l_{13}l_{21}l_{32} > 0.$$

SECTION 8.3. Symmetric, Skew-Symmetric, and Orthogonal Matrices, page 334

Purpose. To introduce the student to the three most important classes of real square matrices and their general properties and eigenvalue theory.

Main Content, Important Concepts

The eigenvalues of a symmetric matrix are real.

The eigenvalues of a skew-symmetric matrix are pure imaginary or zero.

The eigenvalues of an orthogonal matrix have absolute value 1.

Further properties of orthogonal matrices.

Comments on Content

The student should memorize the preceding three statements on the locations of eigenvalues as well as the basic properties of orthogonal matrices (orthonormality of row vectors and of column vectors, invariance of inner product, determinant equal to 1 or -1).

Furthermore, it may be good to emphasize that, since the eigenvalues of an orthogonal matrix may be complex, so may be the eigenvectors. Similarly for skew-symmetric matrices. Both cases are simultaneously illustrated by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{with eigenvectors} \quad \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

corresponding to the eigenvalues i and $-i$, respectively.

Further Comments on the Three Classes of Matrices in This Section

Reality of eigenvalues is a main reason for the importance of symmetric matrices—many quantities in physics, such as mass, energy, etc., are real.

Formula (4) brings in skew-symmetric matrices in a rather natural fashion.

Theorem 3 explains the importance of orthogonal matrices.

Typical examples of the spectra of the matrices considered in this section are illustrated by Probs. 1–10—most importantly, Probs. 4 and 8.

Problems 13–20 should help the student gain a deeper understanding of the concepts and properties of the three classes of matrices considered in this section.

SOLUTIONS TO PROBLEM SET 8.3, page 338

1. Eigenvalues: $3/5 \pm 4/5i$, Eigenvectors: $[i, 1]^T$ and $[-i, 1]^T$ respectively. Skew-symmetric and orthogonal.

2. Eigenvalues $a \pm ib$. Symmetric if $b = 0$; then the eigenvalues are real. Skew-symmetric if $a = 0$; then the eigenvalues are pure imaginary (or zero). Orthogonal if $a^2 + b^2 = 1$; then the eigenvalues have an absolute value of 1.

3. Non-orthogonal; skew-symmetric; Eigenvalues $1 \pm 4i$ with eigenvectors $[-i, 1]^T$ and $[i, 1]^T$.

4. The characteristic equation is

$$\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = \lambda^2 - (2\cos\theta)\lambda + 1 = 0.$$

Hence the eigenvalues are

$$\lambda = \cos\theta \pm i\sin\theta.$$

If $\theta = 0$ (the identity transformation) we have $\lambda = 1$ with multiplicity 2.

If $\theta \neq 0$, we obtain the eigenvectors from

$$x_2 = \mp ix_1, \qquad \text{say,} \qquad [1 \quad \mp i],$$

which are complex; indeed, no (real) direction is preserved under a rotation.

5. Symmetric with eigenvalues $2, -2, 3$ and eigenvectors $[0, 1/\sqrt{3}, 1]^T$, $[0, -\sqrt{3}, 1]^T$, $[1, 0, 0]^T$ respectively; non-orthogonal.

6. $a + 2k$, $[1 \quad 1 \quad 1]^\mathsf{T}$; $a - k$, $[1 \quad 0 \quad -1]^\mathsf{T}$, $[1 \quad -1 \quad 0]^\mathsf{T}$; symmetric (for real a and k)

7. Skew-symmetric; Eigenvalues: $0, \pm\frac{3}{2}i$ with eigenvectors $[-1, -1/2, 1]^T$, $[4/5 + 3/5i, 2/5 - 6/5i, 1]^T$, $[4/5 - 3/5i, 2/5 + 6/5i, 1]^T$ respectively; non-orthogonal.

8. Orthogonal, a rotation about the x_1-axis through an angle θ. Eigenvalues 1 and $\cos\theta \pm i\sin\theta$. Compare with Prob. 4.

9. Skew-symmetric; Eigenvalues: -1, and $\pm i$ with eigenvectors $[0, 1, 0]^T$, $[i, 0, 1]^T$, $[-i, 0, 1]^T$ respectively; orthogonal.

10. Orthogonal; eigenvalues: $-1, 1, 1$, all of absolute value 1. Eigenvectors $[1, -1, 1]^T$, $[-1, 0, 1]^T$, $[1, 1, 0]^T$.

12. CAS Experiment

(a) $\mathbf{A}^{\mathsf{T}} = \mathbf{A}^{-1}$, $\mathbf{B}^{\mathsf{T}} = \mathbf{B}^{-1}$, $(\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} = \mathbf{B}^{-1}\mathbf{A}^{-1} = (\mathbf{AB})^{-1}$. Also $(\mathbf{A}^{-1})^{\mathsf{T}} = (\mathbf{A}^{\mathsf{T}})^{-1} = (\mathbf{A}^{-1})^{-1}$. In terms of rotations it means that the composite of rotations and the inverse of a rotation are rotations.

(b) The inverse is

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

(c) To a rotation of about 36.87°. No limit. For a student unfamiliar with complex numbers this may require some thought.

(d) Limit 0, approach along some spiral.

(e) The matrix is obtained by using familiar values of cosine and sine,

$$\mathbf{A} = \begin{bmatrix} \sqrt{3}/2 & -\frac{1}{2} \\ \frac{1}{2} & \sqrt{3}/2 \end{bmatrix}.$$

16. Let $\mathbf{Ax} = \lambda\mathbf{x}\,(\mathbf{x} \neq \mathbf{0})$, $\mathbf{Ay} = \mu\mathbf{y}\,(\mathbf{y} \neq \mathbf{0})$. Then

$$\lambda\mathbf{x}^{\mathsf{T}} = (\mathbf{Ax})^{\mathsf{T}} = \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} = \mathbf{x}^{\mathsf{T}}\mathbf{A}.$$

Thus

$$\lambda\mathbf{x}^{\mathsf{T}}\mathbf{y} = \mathbf{x}^{\mathsf{T}}\mathbf{Ay} = \mathbf{x}^{\mathsf{T}}\mu\mathbf{y} = \mu\mathbf{x}^{\mathsf{T}}\mathbf{y}.$$

Hence if $\lambda \neq \mu$, then $\mathbf{x}^{\mathsf{T}}\mathbf{y} = 0$, which proves orthogonality.

18. $\det \mathbf{A} = \det(\mathbf{A}^{\mathsf{T}}) = \det(-\mathbf{A}) = (-1)^n \det \mathbf{A} = -\det \mathbf{A} = 0$ if n is odd. Hence the answer is no. For even $n = 2, 4, \cdots$ we have

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad \text{etc,}$$

20. Yes, for instance,

$$\begin{bmatrix} \frac{1}{2} & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

SECTION 8.4. Eigenbases. Diagonalization. Quadratic Forms, page 339

Purpose. This section exhibits the role of bases of eigenvectors ("eigenbases") in connection with linear transformations and contains theorems of great practical importance in connection with eigenvalue problems.

Main Content, Important Concepts

Bases of eigenvectors (Theorems 1, 2)

Similar matrices have the same spectrum (Theorem 3)

Diagonalization of matrices (Theorem 4)

Principal axes transformation of forms (Theorem 5)

Short Courses. Complete omission of this section or restriction to a short look at Theorems 1 and 5.

Comments on Content

Theorem 1 on similar matrices has various applications in the design of numeric methods (Chap. 20), which often use subsequent similarity transformations to tridiagonalize or (nearly) diagonalize matrices on the way to approximations of eigenvalues and eigenvectors. The matrix **X** of eigenvectors [see (5)] also occurs quite frequently in that context.

Theorem 2 is another result of fundamental importance in many applications, for instance, in those methods for numerically determining eigenvalues and eigenvectors. Its proof is substantially more difficult than the proofs given in this chapter.

The theorems in this section give sufficient conditions for the existence of eigenbases (= bases of eigenvectors), namely, the almost trivial Theorem 1 as well as the very important Theorem 2, exhibiting another basic property of symmetric matrices.

This is followed in Theorems 3 and 4 by similarity of matrices and its application to diagonalization.

The second part of the section concerns the principal axes transformation of quadratic forms and its application to conic sections.

The extension of these ideas and results to complex matrices and forms follows in the next section, the last one of this chapter.

SOLUTIONS TO PROBLEM SET 8.4, page 345

2. $\hat{\mathbf{A}} = \begin{bmatrix} -\frac{11}{9} & \frac{1}{18} \\ -\frac{80}{9} & \frac{11}{9} \end{bmatrix}; \quad \lambda = -1, \mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \mathbf{x} = \mathbf{P}\mathbf{y} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}.$

Similarly, for the second eigenvalue we obtain

$$\lambda = 1, \quad y = \begin{bmatrix} 2/9 \\ 1 \end{bmatrix}; \quad \mathbf{x} = \mathbf{P}\mathbf{y} = \begin{bmatrix} -1/9 \\ -40/9 \end{bmatrix}.$$

4. $\hat{\mathbf{A}} = \begin{bmatrix} 15 & 0 & 26 \\ 6 & 3 & 10 \\ -8 & 0 & -14 \end{bmatrix};$ $\begin{aligned} &\lambda = -1, \quad y = [26 \quad 1 \quad -16]^{\mathsf{T}}, \quad x = [4 \quad 1 \quad -2]^{\mathsf{T}} \\ &\lambda = 3, \quad y = [0 \quad 1 \quad 0]^{\mathsf{T}}, \quad x = [0 \quad 1 \quad 0]^{\mathsf{T}} \\ &\lambda = 2, \quad y = [-2 \quad 2 \quad 1]^{\mathsf{T}}, \quad x = [-1 \quad 2 \quad -1]^{\mathsf{T}} \end{aligned}$

5. $\hat{\mathbf{A}} = \begin{bmatrix} 4 & -2 & 4 \\ 0 & -2 & 12 \\ 0 & -2 & 12 \end{bmatrix};$

$\lambda = 10, \mathbf{y} = [1/3, 1, 1]^T, \mathrm{x} = \mathbf{P}\mathbf{y} = [1, 1/3, 1]^T$

$\lambda = 4, \mathbf{y} = [1, 0, 0]^T, \mathrm{x} = \mathbf{P}\mathbf{y} = [0, 1, 0]^T$

$\lambda = 0, \mathbf{y} = [2, 6, 1]^T, \mathrm{x} = \mathbf{P}\mathbf{y} = [6, 2, 1]^T$

6. **Project.** **(a)** This follows immediately from the product representation of the characteristic polynomial of **A**.

(b) $\mathbf{C} = \mathbf{AB}$, $c_{11} = \sum_{l=1}^{n} a_{1l}b_{l1}$, $c_{22} = \sum_{l=1}^{n} a_{2l}b_{l2}$, etc. Now take the sum of these n sums. Furthermore, trace **BA** is the sum of

$$\tilde{c}_{11} = \sum_{m=1}^{n} b_{1m}a_{m1}, \cdots, \tilde{c}_{nn} = \sum_{m=1}^{n} b_{nm}a_{mn},$$

involving the same n^2 terms as those in the double sum of trace **AB**.

(c) By multiplications from the right and from the left we readily obtain

$$\tilde{\mathbf{A}} = \mathbf{P}^2\hat{\mathbf{A}}\mathbf{P}^{-2}.$$

(d) Interchange the corresponding eigenvectors (columns) in the matrix **X** in (5).

9. Eigenvalues: -2, 6; Matrix of corresponding eigenvectors: $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

10. The eigenvalues of **A** are -1 and 1. A matrix of corresponding eigenvectors is

$$\mathbf{X} = \begin{bmatrix} 0 & 1/2 \\ 1 & 1 \end{bmatrix}.$$

$$\mathbf{X}^{-1} = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}.$$

Hence diagonalization gives

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{AX} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

12. **A** has the eigenvalues 10 and -5. A matrix of eigenvectors is

$$\mathbf{X} = \begin{bmatrix} 7 & 11 \\ 13 & -1 \end{bmatrix}.$$

Its inverse is

$$\mathbf{X}^{-1} = \begin{bmatrix} 1/150 & 11/150 \\ 13/150 & -7/150 \end{bmatrix}.$$

Hence diagonalization gives

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{AX} = \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix}.$$

13. Eigenvalues: 2, 9, 6; Matrix of corresponding eigenvectors: $\begin{bmatrix} -\frac{18}{7} & 6/7 & 0 \\ \frac{25}{7} & -6/7 & 1 \\ -1 & 0 & 0 \end{bmatrix}$

14. **A** has the eigenvalues $-2, 4, 1$. A matrix of eigenvectors is

$$\mathbf{X} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$

Its inverse is

$$\mathbf{X}^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -2 & 3 \\ 1 & 2 & -2 \end{bmatrix}.$$

Hence diagonalization gives

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

15. Eigenvalues: 5, -1, 3; Matrix of corresponding eigenvectors: $\begin{bmatrix} 0 & -3 & -1/2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

16. **A** has eigenvalues $-2, 2, 0$. A matrix of corresponding eigenvectors is

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Its inverse is

$$\mathbf{X}^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ -1/2 & 1/2 & 0 \end{bmatrix}.$$

Diagonalization thus gives

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{x} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

18. The symmetric coefficient matrix is

$$\mathbf{C} = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}.$$

It has eigenvalues 5 and -5. Hence the transformed quadratic form is

$$5y_1^2 - 5y_2^2 = 10, \qquad \text{or} \qquad y_1^2 - y_2^2 = 2.$$

This is a hyperbola. The matrix whose columns are normalized eigenvectors of **C** gives the relation between **y** and **x** in the form

$$\mathbf{x} = \begin{bmatrix} 3/10\sqrt{10} & -1/10\sqrt{10} \\ 1/10\sqrt{10} & 3/10\sqrt{10} \end{bmatrix} \mathbf{y}.$$

20. The symmetric coefficient matrix is

$$\mathbf{C} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}.$$

Its eigenvalues are 0 and 10. Hence the transformed form is

$$10y_2^2 = 10.$$

This represents a pair of parallel straight lines

$$10y_2^2 = 10, \qquad \text{thus} \qquad y_2 = \pm 1.$$

The matrix **X** whose columns are normalized eigenvectors of **C** gives the relation between **y** and **x** in the form

$$\mathbf{x} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \mathbf{y}.$$

22. The symmetric coefficient matrix is

$$\mathbf{C} = \begin{bmatrix} 4 & 16 \\ 6 & 13 \end{bmatrix}.$$

Its eigenvalues are 1 and 16. Hence the transformed form is

$$y_1^2 + 16y_2^2 = 16.$$

This represents an ellipse. The matrix whose columns are normalized eigenvectors of **C** gives the relation between **y** and **x** in the form

$$\mathbf{x} = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{y}.$$

24. Transform $Q(\mathbf{x})$ by (9) to the canonical form (10). Since the inverse transform $\mathbf{y} = \mathbf{X}^{-1}\mathbf{x}$ of (9) exists, there is a one-to-one correspondence between all $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$. Hence the values of $Q(\mathbf{x})$ for $\mathbf{x} \neq \mathbf{0}$ coincide with the values of (10) on the right. But the latter are obviously controlled by the signs of the eigenvalues in the three ways stated in the theorem. This completes the proof.

SECTION 8.5. Complex Matrices and Forms. *Optional,* page 346

Purpose. This section is devoted to the three most important classes of complex matrices and corresponding forms and eigenvalue theory.

Main Content, Important Concepts

Hermitian and skew-Hermitian matrices

Unitary matrices, unitary systems

Location of eigenvalues (Fig. 163)

Quadratic forms, their symmetric coefficient matrix

Hermitian and skew-Hermitian forms

Background Material. Section 8.3, which the present section generalizes. The prerequisites on complex numbers are very modest, so that students will need hardly any extra help in that respect.

Short Courses. This section can be omitted.

The importance of these matrices results from quantum mechanics as well as from mathematics itself (e.g., from unitary transformations, product representations of nonsingular matrices $\mathbf{A} = \mathbf{UH}$, $\mathbf{U}$ unitary, $\mathbf{H}$ Hermitian, etc.).

The determinant of a unitary matrix (see Theorem 4) may be complex. For example, the matrix

$$\mathbf{A} = \frac{1+i}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is unitary and has

$$\det \mathbf{A} = i.$$

Comments on Problems

Complex matrices appear in quantum mechanics; see Prob. 7, etc.

Problems 13–20 give an impression of calculations for complex matrices.

Normal matrices, defined in Prob.18, play an important role in a more extended theory of complex matrices.

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1. Hermitian; Eigenvalues: 3, 1; and the corresponding matrix of eigenvectors is $\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$

2. **Skew-Hermitian.** Eigenvalues and eigenvectors are

$$-i, [-1 + i \quad 2]^{\mathsf{T}}$$

and

$$2i, [2 \quad 1 + i]^{\mathsf{T}}.$$

3. Non-Hermitian; Eigenvalues: $\frac{1}{4} \pm i\sqrt{2}$, and the matrix of eigenvectors is $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

4. Skew-Hermitian, as well as unitary, eigenvalues i and $-i$, eigenvectors $[1 \quad 1]^{\mathsf{T}}$ and $[1 \quad -1]^{\mathsf{T}}$, respectively.

5. Non-Hermitian; Eigenvalues: i, $-i$, $-i$, and the corresponding matrix of eigenvectors is

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. Hermitian. Eigenvalues and eigenvectors are

$$\begin{aligned} &-4, [i \quad -1-i \quad 1]^{\mathsf{T}} \\ &0, [1 \quad 0 \quad i]^{\mathsf{T}} \\ &4, [i \quad 1+i \quad 1]^{\mathsf{T}}. \end{aligned}$$

8. Eigenvectors are as follows. (Multiplication by a complex constant may change them drastically!)

$$\begin{array}{lll} \text{For } \mathbf{A} & [1-3i \quad 5]^{\mathsf{T}}, & [1-3i \quad -2]^{\mathsf{T}} \\ \text{For } \mathbf{B} & [2+i \quad i]^{\mathsf{T}}, & [2+i \quad -5i]^{\mathsf{T}} \\ \text{For } \mathbf{C} & [1 \quad 1]^{\mathsf{T}}, & [1 \quad -1]^{\mathsf{T}}. \end{array}$$

9. Skew-Hermitian; $\bar{\mathbf{x}}^T\mathbf{A}\mathbf{x} = 8 - 12i$.

10. The matrix is non-Hermitian.

$$\bar{\mathbf{x}}^T\mathbf{A}\mathbf{x} = [3, -2i][-4 + i, 3 + 6i]^T = -3i$$

11. Skew-Hermitian; $\bar{\mathbf{x}}^T\mathbf{A}\mathbf{x} = -4i$.

12. The matrix is Hermitian. We obtain the real value

$$[1 \quad -i \quad i]\mathbf{A}[1 \quad i \quad -i]^{\mathsf{T}} = [1 \quad -i \quad i][-4i \quad 2i \quad 4-2i]^{\mathsf{T}} = 4.$$

14. $(\overline{\mathbf{BA}})^{\mathsf{T}} = (\overline{\mathbf{B}}\overline{\mathbf{A}})^{\mathsf{T}} = \overline{\mathbf{A}}^{\mathsf{T}}\overline{\mathbf{B}}^{\mathsf{T}} = \mathbf{A}(-\mathbf{B}) = -\mathbf{AB}$. For the matrices in Example 2.

$$\mathbf{AB} = \begin{bmatrix} 1 + 19i & 5 + 3i \\ -23 + 10i & -1 \end{bmatrix}.$$

16. The inverse of a product $\mathbf{UV}$ of unitary matrices is

$$(\mathbf{UV})^{-1} = \mathbf{V}^{-1}\mathbf{U}^{-1} = \overline{\mathbf{V}}^{\mathsf{T}}\overline{\mathbf{U}}^{\mathsf{T}} = (\overline{\mathbf{UV}})^{\mathsf{T}}.$$

This proves that $\mathbf{UV}$ is unitary.

We show that the inverse $\mathbf{A}^{-1} = \mathbf{B}$ of a unitary matrix $\mathbf{A}$ is unitary. We obtain

$$\mathbf{B}^{-1} = (\mathbf{A}^{-1})^{-1} = (\overline{\mathbf{A}}^{\mathsf{T}})^{-1} = (\overline{\mathbf{A}^{-1}})^{\mathsf{T}} = \overline{\mathbf{B}}^{\mathsf{T}},$$

as had to be shown.

18. $\overline{\mathbf{A}}^{\mathsf{T}}\mathbf{A} = \mathbf{A}^2 = \mathbf{A}\overline{\mathbf{A}}^{\mathsf{T}}$ if $\mathbf{A}$ is Hermitian, $\overline{\mathbf{A}}^{\mathsf{T}}\mathbf{A} = -\mathbf{A}^2 = \mathbf{A}(-\mathbf{A}) = \mathbf{A}\overline{\mathbf{A}}^{\mathsf{T}}$ if $\mathbf{A}$ is skew-Hermitian, $\overline{\mathbf{A}}^{\mathsf{T}}\mathbf{A} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{A}\overline{\mathbf{A}}^{\mathsf{T}}$ if $\mathbf{A}$ is unitary.

20. For instance,

$$\begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix}$$

is not normal. A normal matrix that is not Hermitian, skew-Hermitian, or unitary is obtained if we take a unitary matrix and multiply it by 2 or some other real factor different from ± 1.

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11. Eigenvalues: 1, 2, and the corresponding matrix of eigenvectors is $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

12. The eigenvalues are -1 and 1. Corresponding eigenvectors are $[2 \quad 3]^{\mathsf{T}}$ and $[1 \quad 2]^{\mathsf{T}}$, respectively. Note that this basis is not orthogonal.

13. Eigenvalues: 11/2, 13/2, and the matrix of eigenvectors is $\begin{bmatrix} 2/5 & 2/3 \\ 1 & 1 \end{bmatrix}$

14. One of the eigenvalues is 9. Its algebraic and geometric multiplicities are 2. Corresponding linearly independent eigenvectors are $[1 \quad 0 \quad -2]^{\mathsf{T}}$ and $[0 \quad 1 \quad 2]^{\mathsf{T}}$. The other eigenvalue is 4.5. A corresponding eigenvector is $[2 \quad -2 \quad 1]^{\mathsf{T}}$.

15. Eigenvalues: $\pm 12i$, 0, and the matrix of eigenvectors is

$$\begin{bmatrix} -1/4 + 3/4i & -1/4 - 3/4i & 2 \\ 1/4 + 3/4i & 1/4 - 3/4i & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

16. The eigenvalues of $\mathbf{A}$ are -17 and 9. The similar matrix, having the same eigenvalues, is

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 9 & 17 \\ 9 & -17 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & -17 \end{bmatrix}$$

17. Eigenvalues: -5 and 7. $\hat{\mathbf{A}} = \begin{bmatrix} \frac{35}{2} & -\frac{35}{2} \\ \frac{27}{2} & -\frac{31}{2} \end{bmatrix}$

18. $\mathbf{A}$ has the eigenvalues $-2, -1, 2$. The inverse of $\mathbf{P}$ is

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & -8 & 31 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

The similar matrix $\hat{\mathbf{A}}$, having the same eigenvalues, is

$$\hat{\mathbf{A}} = \begin{bmatrix} 1 & -8 & 31 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & -26 & 52 \\ 0 & 2 & 6 \\ -1 & -7 & 11 \end{bmatrix} = \begin{bmatrix} -35 & -259 & 345 \\ 3 & 23 & -27 \\ -1 & -7 & 11 \end{bmatrix}.$$

19. Eigenvalues: $2/5$ and $-1/20$; The corresponding matrix of eigenvectors is

$$\begin{bmatrix} 4/5 & 5/4 \\ 1 & 1 \end{bmatrix} \text{ and its inverse is } \begin{bmatrix} -\frac{20}{9} & \frac{25}{9} \\ \frac{20}{9} & -\frac{16}{9} \end{bmatrix}$$

20. Eigenvalues are 26 and 316. The corresponding matrix of eigenvalues is

$$\mathbf{X} = \begin{bmatrix} -1/17 & 17 \\ 1 & 1 \end{bmatrix}.$$

Note that the vectors are orthogonal. Its inverse is

$$\mathbf{X}^{-1} = \begin{bmatrix} -\frac{17}{290} & \frac{289}{290} \\ \frac{17}{290} & \frac{1}{290} \end{bmatrix}.$$

Diagonalization gives

$$\begin{bmatrix} -\frac{17}{290} & \frac{289}{290} \\ \frac{17}{290} & \frac{1}{290} \end{bmatrix} \begin{bmatrix} -\frac{316}{17} & 442 \\ 316 & 26 \end{bmatrix} = \begin{bmatrix} 316 & 0 \\ 0 & 26 \end{bmatrix}.$$

22. The symmetric coefficient matrix is

$$\mathbf{C} = \begin{bmatrix} 9 & -3 \\ -3 & 17 \end{bmatrix}.$$

Its eigenvalues are 8 and 18; they are both positive real. The transformed form is

$$8y_1^2 + 18y_2^2 = 36.$$

This is the canonical form; there is no y_1y_2-term. It represents an ellipse. The matrix **X** whose columns are normalized eigenvectors of **C** gives the relationship between **y** and **x** in the form

$$\mathbf{x} = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{y}.$$

24. The symmetric coefficient matrix is

$$\mathbf{C} = \begin{bmatrix} 6 & 8 \\ 8 & -6 \end{bmatrix}.$$

Its eigenvalues are 10 and -10. The transformed form is

$$10y_1^2 - 10y_2^2 = 10(y_1 + y_2)(y_1 - y_2) = 0.$$

It represents two perpendicular straight lines through the origin. The matrix **X** whose columns are normalized eigenvectors of **C** gives the relationship between **x** and **y** in the form

$$\mathbf{x} = \begin{bmatrix} 2/5\sqrt{5} & -1/5\sqrt{5} \\ 1/5\sqrt{5} & 2/5\sqrt{5} \end{bmatrix} \mathbf{y}.$$